On the Capacity Region for Index Coding

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***Abstract*—A new inner bound on the capacity region of a general index coding problem is established. Unlike most existing bounds that are based on graph theoretic or algebraic tools, the bound is built on a random coding scheme and optimal decoding, and has a simple polymatroidal single-letter expression. The utility of the inner bound is demonstrated by examples that include the capacity region for all index coding problems with up to five messages (there are 9846 nonisomorphic ones).**

I. Introduction

Consider the simple communication problem in Figure 1, which is often referred to as the *index coding* problem. The sender wishes to communicate N messages $M_j \in [1:2^{nR_j}]$, $j \in [1:N]$, to their respective receivers over a common noiseless link that carries n bits X^n. Each receiver $j \in [1:N]$ has prior knowledge of $M_{\mathcal{A}_j}$, i.e., a subset $\mathcal{A}_j \subseteq [1:N] \setminus \{j\}$ of the messages. Based on this side information $M_{\mathcal{A}_j}$ and the received bits X^n, receiver j finds the estimate $\hat{M}_j$ of the message M_j. A nontrivial tradeoff arises between the rates R_j, $j \in [1:N]$, of the messages since they compete for the shared broadcast medium for receivers with incompatible knowledge.

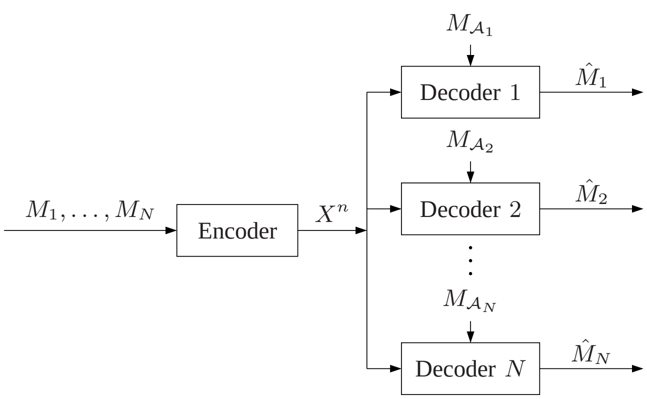

Fig. 1. The index coding problem.

We define a $(2^{nR_1}, \dots, 2^{nR_N}, n)$ code for index coding by an encoder $x^n(m_1, \dots, m_N)$ and N decoders $\hat{m}_j(x^n, m_{\mathcal{A}_j})$, $j \in [1:N]$. We assume that the message tuple $(M_1, \dots, M_N)$ is uniform over $[1:2^{nR_1}] \times \cdots \times [1:2^{nR_N}]$, that is, the messages are uniform and independent of each other. The average probability of error is then defined as $P_e^{(n)} = \mathsf{P}\{(\hat{M}_1, \dots, \hat{M}_N) \neq (M_1, \dots, M_N)\}$. A rate tuple $(R_1, \dots, R_N)$ is said to be achievable if there exists a sequence of $(2^{nR_1}, \dots, 2^{nR_N}, n)$ codes such that $\lim_{n\to\infty} P_e^{(n)} = 0$. The capacity region $\mathscr{C}$ of the index coding problem is the closure of the set of achievable rate tuples $(R_1, \dots, R_N)$. The goal is to find the capacity region and the optimal coding scheme that achieves it.

Note that an index coding problem is fully characterized by the side information sets $\mathcal{A}_j$, $j \in [1:N]$. As an example, consider the 3-message index coding problem with $\mathcal{A}_1 = \{2\}$, $\mathcal{A}_2 = \{1,3\}$, and $\mathcal{A}_3 = \{1\}$. We represent this problem compactly as

$$(1|2),\ (2|1,3),\ (3|1), \tag{1}$$

or as a directed graph (see Figure 2(a)), where nodes represent indices of the messages/receivers and edges represent availability of side information (e.g., the edge $1 \to 2$ means that side information M_1 is available at receiver 2). Note that this 3-message index coding problem can be represented as an instance of the network coding problem [1] as illustrated in Figure 2(b). The same observation can be made for any index coding problem; thus, index coding is a special case of network coding.

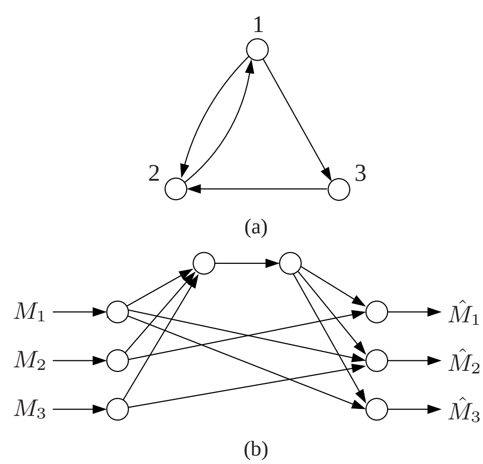

Fig. 2. (a) A directed graph representation. (b) The equivalent network coding problem. Here every edge of the graph can carry up to 1 bit per transmission.

First introduced by Birk and Kol [2] in the context of satellite broadcast communication, the index coding problem has been studied extensively over the past six years in the theoretical computer science and network coding communities with many exciting contributions of combinatorial and algebraic flavors (see, for example, [3]–[15] and the references

therein). Our Shannon-theoretic formulation of the problem closely follows that of Maleki, Cadambe, and Jafar [16], who established the capacity region for several interesting classes of index coding problems using interference alignment [17]. Despite all these developments, the capacity region of a general index coding problem is not known.

Confirming Maslow's axiom [18] "if all you have is a hammer, everything looks like a nail," we propose a *random coding* approach, replacing more advanced coding schemes of an algebraic nature. This approach is more in the spirit of the original paper by Ahlswede, Cai, Li, and Yeung [1], where random coding (binning) was used to establish the network coding theorem. In particular, we develop a *composite coding* scheme based on random coding and establish a corresponding single-letter inner bound on the capacity region.

Instead of mechanical proofs, this paper focuses on basic intuitions behind our coding scheme, which we develop gradually from simpler coding schemes—"flat coding" in Section III and "dual index coding" in Section IV. The composite coding scheme is explained in Section V. The next section discusses known outer bounds on the capacity region.

II. Outer Bounds

We first recall the following outer bound on the capacity region, which is a simple consequence of Fano's inequality and the submodularity of entropy; see, for example, [9] (or [19] for a similar bound in the context of a general network coding problem).

Theorem 1: Let $\mathcal{B}_j = [1:N] \setminus (\{j\} \cup \mathcal{A}_j)$ be the index set of interfering messages. If $(R_1, \ldots, R_N)$ is achievable, then it must satisfy

$$R_j \le T_{\{j\}\cup\mathcal{B}_j} - T_{\mathcal{B}_j}, \quad j \in [1:N],$$

for some $T_{\mathcal{J}}$, $\mathcal{J} \subseteq [1:N]$, such that

1) $T_\emptyset = 0$,
2) $T_{[1:N]} = 1$,
3) for all $\mathcal{J} \subseteq \mathcal{K}$, $T_{\mathcal{J}} \le T_{\mathcal{K}}$, and
4) for all $\mathcal{J}$ and $\mathcal{K}$, $T_{\mathcal{J}\cap\mathcal{K}} + T_{\mathcal{J}\cup\mathcal{K}} \le T_{\mathcal{J}} + T_{\mathcal{K}}$.

It is not known whether this outer bound is tight in general. Sometimes a relaxed version of the bound is handy.

Corollary 1: Let $\mathcal{G} = ([1:N], \mathcal{E})$ be a directed graph representation of the index coding problem $(j|\mathcal{A}_j)$, $j \in [1:N]$, that is, $\mathcal{V} = [1:N]$ and $(j,k) \in \mathcal{E}$ iff $j \in \mathcal{A}_k$. If $(R_1, \ldots, R_N)$ is achievable, then it must satisfy

$$\sum_{j\in\mathcal{J}} R_j \le 1$$

for all $\mathcal{J} \subseteq [1:N]$ such that the subgraph of $\mathcal{G}$ over $\mathcal{J}$ does not contain a directed cycle.

The following example, due to [14], [16], illustrates that the two outer bounds do not coincide in general.

Example 1: Consider the symmetric five-message index coding problem $(j\,|\,j-1, j+1)$, $j \in [1:5]$, namely,

$$(1|5,2),\ (2|1,3),\ (3|2,4),\ (4|3,5),\ (5|4,1).$$

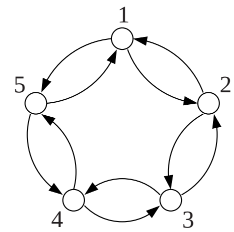

Fig. 3. A graph representation of the 5-message index coding problem.

The corresponding graph representation is depicted in Figure 3. Applying Corollary 1, we obtain

$$\begin{aligned} R_1 + R_3 &\le 1,\\ R_2 + R_4 &\le 1,\\ R_3 + R_5 &\le 1,\\ R_4 + R_1 &\le 1,\\ R_5 + R_2 &\le 1. \end{aligned} \tag{2}$$

In comparison, Theorem 1 leads to the inequality

$$R_1 + R_2 + R_3 + R_4 + R_5 \le 2, \tag{3}$$

in addition to the above five inequalities. As we discuss in Section V, the resulting six inequalities characterize the capacity region of the index coding problem.

III. Flat Coding

Consider the following simple random coding scheme. For each $(m_1, \ldots, m_N) \in [1:2^{nR_1}] \times \cdots \times [1:2^{nR_N}]$, generate a codeword $x^n(m_1, \ldots, m_N)$ randomly and independently as a Bern(1/2) sequence. To communicate $(m_1, \ldots, m_N)$, the sender transmits $x^n = x^n(m_1, \ldots, m_N)$. Receiver j uses simultaneous nonunique decoding [20] and finds the unique $\hat{m}_j \in [1:2^{nR_j}]$ such that $x^n(\hat{m}_j, m_{\mathcal{A}_j}, m_{\mathcal{B}_j})$ is jointly typical with (i.e., identical to) the received sequence x^n for some $m_{\mathcal{B}_j}$, where $\mathcal{B}_j = [1:N] \setminus (\{j\} \cup \mathcal{A}_j)$. Since the codebook generation is "flat" (compared with "layered" superposition coding), simultaneous nonunique decoding is essentially identical to performing the unique decoding of $(\hat{m}_j, \hat{m}_{\mathcal{B}_j})$ and then discarding the unnecessary part $\hat{m}_{\mathcal{B}_j}$.

This "flat coding" scheme achieves the following inner bound.

Proposition 1: A rate tuple $(R_1, \ldots, R_N)$ is achievable for the index coding problem $(j|\mathcal{A}_j)$, $j \in [1:N]$, if

$$R_j + \sum_{k\in\mathcal{B}_j} R_k < 1, \quad j \in [1:N].$$

As an example, consider the 3-message problem in (1). Under flat coding, receiver 1 finds the unique $\hat{m}_1$ such that $x^n(\hat{m}_1, m_2, m_3) = x^n$ for some $m_3 \in [1:2^{nR_3}]$ and the given side information m_2. By the packing lemma [21, Sec. 3.4], it can be readily shown that the probability of decoding error for receiver 1 tends to zero as $n \to \infty$ if

$$R_1 + R_3 < 1. \tag{4}$$

Similarly, we obtain $R_2 < 1$ (inactive) and

$$R_2 + R_3 < 1. \tag{5}$$

By comparing with Theorem 1 (or Corollary 1), it can be easily checked that the rate region formed by (4) and (5) is indeed the capacity region.

It can be easily verified that for all index coding problems with 1, 2, and 3 messages—there are 1, 3, and 16 nonisomorphic problems [22]—flat coding achieves the capacity region. More generally, among 218 four-message index coding problems, time sharing of flat coding over subsets of messages achieves the capacity region for all but three. The following is one of the three exceptions.

Example 2: Consider the 4-message index coding problem

$$(1|4),\ (2|3,4),\ (3|1,2),\ (4|2,3).$$

On the one hand, flat coding yields an inner bound on the capacity region that consists of the rate quadruples (R_1, R_2, R_3, R_4) such that

$$\begin{aligned} R_1 + R_2 + R_3 &< 1, \\ R_1 + R_4 &< 1, \\ R_3 + R_4 &< 1. \end{aligned}$$

It can be verified that this inner bound cannot be improved upon by time sharing over subsets. On the other hand, Theorem 1 yields an outer bound that consists of the rate quadruples (R_1, R_2, R_3, R_4) such that

$$\begin{aligned} R_1 + R_2 &\le 1, \\ R_1 + R_3 &\le 1, \\ R_1 + R_4 &\le 1, \\ R_3 + R_4 &\le 1. \end{aligned} \tag{6}$$

We will see in Section V that this outer bound is tight.

While flat coding is suboptimal in general, the analysis (i.e., the proof of Proposition 1) is trivial and does not rely on any graph theoretic machinery. This observation will become crucial when we generalize the coding scheme subsequently.

IV. Dual Index Coding

Before we move on to a more powerful random coding scheme, we introduce a communication problem (depicted in Figure 4) that is, in some sense, dual to the index coding problem. Here a set of $(2^N - 1)$ senders wish to communicate a message tuple $(M_1, \ldots, M_N)$ to a common receiver, each encoding a subtuple $M_{\mathcal{J}}$ into a separate index $W_{\mathcal{J}} \in [1 : 2^{nS_{\mathcal{J}}}]$ for $\mathcal{J} \subseteq [1 : N]$ nonempty. What is the capacity region (as a function of the rates $S_{\mathcal{J}}$)?

This problem turns out to be a special case of the general multiple access channel (MAC) with correlated messages studied by Han [23]. For the general MAC, superposition coding achieves the capacity region that is characterized by independent auxiliary random variables $U_1, \ldots, U_N$, each corresponding to a message. However, for the dual index coding problem, we can characterize the capacity region explicitly.

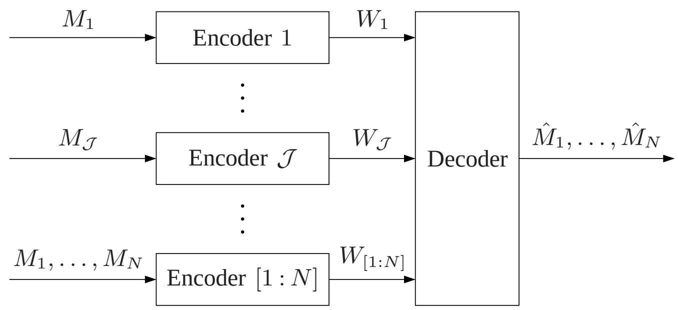

Fig. 4. The dual index coding problem.

Proposition 2: The capacity region of the dual index coding problem is the set of rate tuples $(R_1, \ldots, R_N)$ such that

$$\sum_{j \in \mathcal{J}} R_j \le \sum_{\mathcal{J}' \subseteq [1:N] : \mathcal{J}' \cap \mathcal{J} \neq \emptyset} S_{\mathcal{J}'} \tag{7}$$

for all $\mathcal{J} \subseteq [1 : N]$.

What is perhaps more important than this explicit characterization of the capacity region is the fact that it can be achieved by flat coding, which we will utilize later.

As an example, consider the three-message three-sender dual index coding problem in Figure 5, where $S_{1,2} = 1$ and $S_{1,3} = S_{1,2,3} = 2$. By (7), the capacity region is the set of rate triples (R_1, R_2, R_3) such that

$$\begin{aligned} R_1 + R_2 + R_3 &\le 5, \\ R_2 &\le 3, \\ R_3 &\le 4. \end{aligned} \tag{8}$$

This can be achieved via flat coding of (M_1, M_2), (M_1, M_3), and (M_1, M_2, M_3), respectively, and simultaneous decoding at the receiver.

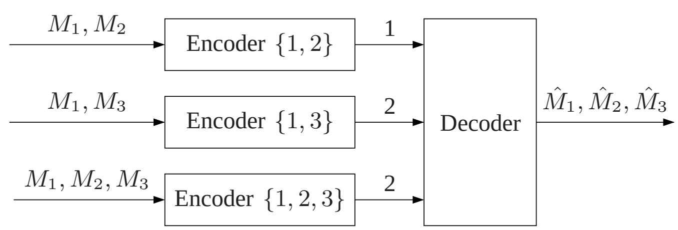

Fig. 5. An example for dual index coding.

V. Composite coding

Equipped with the results in the previous two subsections, we now introduce another random coding scheme, which we refer to as *composite coding*. This is best described by an example.

We revisit the 5-message problem [14], [16] in Example 1. As the first step of composite coding, the sender encodes (M_1, M_2) into an index $W_{1,2}$ at rate $S_{1,2}$ using random coding, and similarly encodes (M_2, M_3), (M_3, M_4), (M_4, M_5), and (M_1, M_5), respectively, into indices $W_{2,3}$, $W_{3,4}$, $W_{4,5}$,

and $W_{1,5}$. Equivalently, we decompose the sender into 5 "virtual" senders, each mapping one of the above pairs of messages (as in the dual index coding problem). As the second step, the sender uses flat coding to communicate the "composite" indices $W_{1,2}, W_{2,3}, W_{3,4}, W_{4,5}, W_{1,5}$. As with encoding, decoding also takes two steps. Each receiver first recovers the composite indices and then recovers the desired message from the composite indices. For example, receiver 1 recovers $W_{1,2}, W_{1,5}$ (and other composite indices too). Now since it has side information (M_2, M_5), it can recover M_1 from $(W_{1,2}, W_{1,5})$ at rate $S_{1,2} + S_{1,5}$. Following similar steps for other receivers and incorporating the flat coding rate condition, it can be easily verified that the rate quintuple $(R_1, R_2, R_3, R_4, R_5)$ is achievable if

$$\begin{aligned}
R_1 &< S_{1,2} + S_{1,5}, \\
R_2 &< S_{1,2} + S_{2,3}, \\
R_3 &< S_{2,3} + S_{3,4}, \\
R_4 &< S_{3,4} + S_{4,5}, \\
R_5 &< S_{1,5} + S_{4,5}
\end{aligned}$$

for some $(S_{1,2}, S_{2,3}, S_{3,4}, S_{4,5}, S_{1,5})$ such that $S_{1,2} + S_{2,3} + S_{3,4} + S_{4,5} + S_{1,5} \leq 1$. Fourier–Motzkin eliminiation [21, Appendix D] of the composite index rates yields the inequalities in (2) and (3) in the outer bound, establishing the capacity region.

As another example, we revisit the four-message problem in Example 2. In this case, we use the composite indices $W_{1,4}$ and $W_{1,2,3,4}$ of rates $S_{2,3}$ and $S_{1,2,3,4}$, respectively. Then, it can be easily verified from Proposition 2 that receiver 1 can recover M_1 if $R_1 < S_{1,4}$; receiver 2 can recover M_2 (and M_1 as well) if $R_1 + R_2 < S_{1,4} + S_{1,2,3,4}$ and $R_2 < S_{1,2,3,4}$; receiver 3 can recover M_3 (and M_4 as well) if $R_3 + R_4 < S_{1,4} + S_{1,2,3,4}$ and $R_3 < S_{1,2,3,4}$; and receiver 4 can recover M_4 (and M_1 as well) if $R_1 + R_4 < S_{1,4} + S_{1,2,3,4}$. By eliminating $S_{1,4}$ and $S_{1,2,3,4}$ under the constraint $S_{1,4} + S_{1,2,3,4} \leq 1$, we obtain the same set of inequalities as in the outer bound (6), establishing the capacity region.

In general, we can utilize $(2^N - 1)$ virtual senders to encode N messages. Moreover, the receivers can employ simultaneous nonunique decoding for the second-step decoding (or equivalently, ignore some composite indices in an optimal manner). This coding scheme is illustrated in Figure 6.

To characterize the performance of the composite coding scheme, for each $(S_{\mathcal{J}} \colon \mathcal{J} \subseteq [1:N])$, we define the polymatroidal region $\mathscr{R}(\mathcal{K} \,|\, \mathcal{K}')$ as the set of rate tuples $(R_1, \ldots, R_N)$ such that

$$\sum_{j \in \mathcal{J}} R_j < \sum_{\mathcal{J}' \subseteq \mathcal{K} \cup \mathcal{K}' \colon \mathcal{J}' \cap \mathcal{J} \neq \emptyset} S_{\mathcal{J}'} \tag{9}$$

for all $\mathcal{J} \subseteq \mathcal{K} \setminus \mathcal{K}'$. This region corresponds to the capacity region of the dual index coding problem (Proposition 2) for the desired message set $\mathcal{K}$ with side information $\mathcal{K}'$. We are now ready to state the main result of the paper.

Theorem 2 (Composite-coding inner bound): A rate tuple $(R_1, \ldots, R_N)$ is achievable for the index coding problem $(j|\mathcal{A}_j)$, $j \in [1:N]$, if

$$(R_1, \ldots, R_N) \in \bigcap_{j \in [1:N]} \bigcup_{\mathcal{K} \subseteq [1:N] \colon j \in \mathcal{K}} \mathscr{R}(\mathcal{K} \,|\, \mathcal{A}_j) \tag{10}$$

for some $(S_{\mathcal{J}} \colon \mathcal{J} \subseteq [1:N])$ such that $\sum_{\mathcal{J} \colon \mathcal{J} \not\subseteq \mathcal{A}_j} S_{\mathcal{J}} \leq 1$ for all $j \in [1:N]$.

At a first glance, composite coding seems to be time sharing of flat coding over all subsets of $[1:N]$. However, it employs the optimal decoding rule that utilizes all composite indices (subsets) that are relevant to the desired message. As such, the corresponding rate region has a very similar form as the optimal rate region for interference networks with random coding [24].

Using the polco tool for polyhedral computations [25], we have computed the composite-coding inner bound and the outer bound in Theorem 1 for all 9608 nonisomorphic five-message index coding problems [22]. In all cases, both bounds agree, establishing the capacity region.

To further demonstrate the utility of composite coding, we revisit the following example in [16].

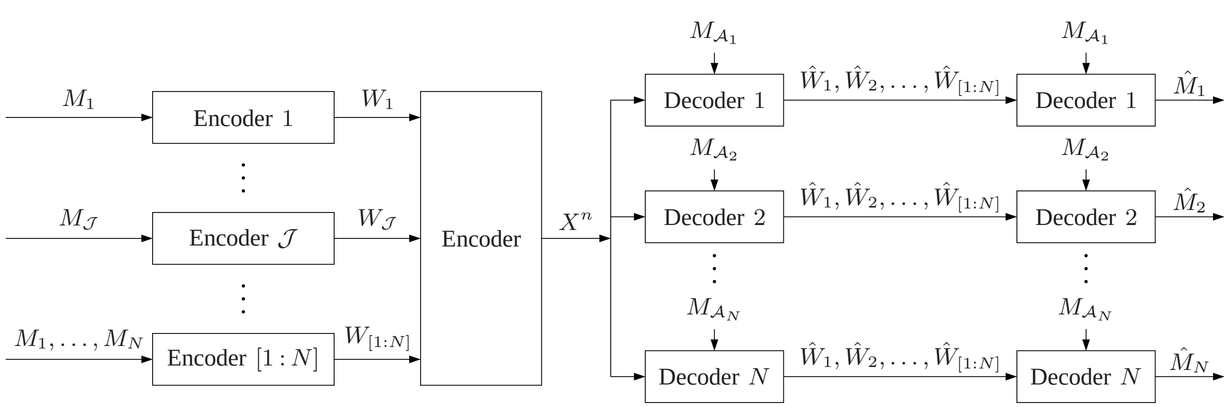

Fig. 6. Composite coding scheme.

Example 3: Consider the N-message *symmetric* index coding problem

$$(j \mid j-D, j-D+1, \ldots, j-1, j+1, \ldots, j+U)$$

for $j \in [1:N]$. For example, the 5-message problem in Example 1 is a special case of this problem with $N = 5$ and $D = U = 1$. We assume without loss of generality that $0 \le U \le D \le N-U-1$. Set $S_{\mathcal{J}} = 1/(N-(D-U))$ if $\mathcal{J}$ is of the form $[k : k+U]$. Set $S_{\mathcal{J}} = 0$ otherwise. Since receiver $j \in [1:N]$ has $M_{j+1}, \ldots, M_{j+D}$ as side information, it already knows the $(D-U)$ composite indices $W_{[j+1:j+1+U]}, \ldots, W_{[j+D-U:j+D]}$. Thus, there are only $N-(D-U)$ composite indices that needs to be recovered from x^n, which is feasible since $\sum_{\mathcal{J}:\mathcal{J}\not\subseteq\mathcal{A}_j} S_{\mathcal{J}} = 1$. Now receiver j can recover M_j from the composite indices $W_{[j-U:j]}, \ldots, W_{[j:j+U]}$, provided that

$$R_j < S_{[j-U:j]} + \cdots + S_{[j:j+U]}.$$

Hence, the symmetric rate of $(U+1)/(N-D+U)$ is achievable. In [16] it is shown that this symmetric rate is in fact optimal, which can be also verified directly by the outer bound in Theorem 1. For $N = 6, U = 1$, and $D = 2$, that is,

$$(1|2,5,6), (2|1,3,6), (3|1,2,4),$$
$$(4|2,3,5), (5|3,4,6), (6|1,4,5),$$

the symmetric rate of $2/5$ is optimal. In fact, simplifying Theorems 1 and 2 yields the capacity region that consists of the rate sextuple $(R_1, \ldots, R_6)$ such that

$$\begin{aligned} R_j + R_{j+2} &\le 1, \quad j \in [1:6], \\ R_j + R_{j+3} &\le 1, \quad j \in [1:6], \\ R_j + R_{j+1} + R_{j+2} + R_{j+3} + R_{j+4} &\le 2, \quad j \in [1:6]. \end{aligned}$$

In particular, this region is achievable by using composite indices W_1, W_2, W_3, W_4, W_5, W_6, $W_{1,2}$, $W_{2,3}$, $W_{3,4}$, $W_{4,5}$, $W_{5,6}$, and $W_{1,6}$.

VI. Concluding Remarks

Based on a first principle in Shannon's random coding, this paper has established the composite-coding inner bound on the general index coding problem. This inner bound is simple, easy to compute, yet is powerful and tight for all index coding problems of up to five messages as well as many existing examples. In a sense, random coding is a "jackknife" rather than a "hammer."

The polymatroidal structure of the composite-coding inner bound and the submodularity of the outer bound suggest a deeper connection rooted in matroid theory [19], [26]. In addition to evaluating the inner and outer bounds for more examples (there are 1540944 nonisomorphic six-message index coding problems), future studies will focus on analyzing the algebraic structures of these bounds to investigate what lies in the path to establishing the capacity region of a general index coding problem.

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